§4.4, 5.5 More integrals

① Indefinite Integral of f: $\int f(x)\,dx = F(x) + C$, where $F'(x) = f(x)$.

- We want a new notation for the most general anti-D of $f(x)$,

  $\int f(x)\,dx = F(x) + C$, where $F(x)$ is any anti-D of $f(x)$, i.e., $F'(x) = f(x)$.

- In particularly, $\int F'(x)\,dx = F(x) + C$, and $F(x) = \int F'(x)\,dx$ $(+C)$

☆ • Most important integrals: $\int x^n\,dx = \frac{1}{n+1} x^{n+1} + C$, $n \neq -1$. $\boxed{\int 1\,dx = x + C}$

☆ $\int \sin x\,dx = -\cos x + C$, $\int \cos x\,dx = \sin x + C$, $\int \sec^2 x\,dx = \tan x + C$, $\int \tan x \cdot \sec x\,dx = \sec x + C$.

eg.1. Find the most general function F such that $F'(x) = \frac{4}{\sqrt[3]{x}} - 2\tan x \cdot \sec x$.

s/n: $F(x) = \int \frac{4}{\sqrt[3]{x}} - 2\tan x \cdot \sec x\,dx$

$= \int 4 \cdot x^{-\frac{1}{3}} - 2\tan x \cdot \sec x\,dx$ Apply the formula of $\int x^n\,dx$ with $n = -\frac{1}{3}$

$= 4 \cdot \int x^{-\frac{1}{3}}\,dx - 2 \cdot \int \tan x \cdot \sec x \cdot dx$

$= \boxed{4 \cdot \frac{1}{-\frac{1}{3}+1} \cdot x^{-\frac{1}{3}+1} - 2 \cdot \sec x + C} = \frac{2}{3} x^{\frac{2}{3}} - 2 \cdot \sec x + C$

eg2. Evaluate the indefinite integral

$\int u \cdot (\sqrt{u} + 2)\,du = \int u \cdot u^{\frac{1}{2}} + 2u\,du$

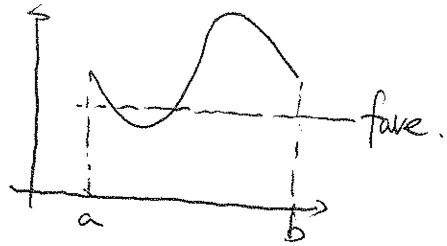
$= \int u^{\frac{3}{2}} + 2 \cdot u\,du$ Apply the formula with $n = \frac{3}{2}$ and $n = 1$

$= \frac{1}{\frac{3}{2}+1} \cdot u^{\frac{3}{2}+1} + 2 \cdot \frac{1}{1+1} u^{1+1} + C$

$= \boxed{\frac{2}{5} \cdot u^{\frac{5}{2}} + u^2 + C}$

- Average of $f(x)$ over $[a,b]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



eg.4 Find the average of $f(x)=\frac{1}{x^2}$ on the interval $[1,4]$

sln: $$f_{ave} = \frac{1}{4-1}\int_1^4 \frac{1}{x^2}\,dx = \frac{1}{3}\int_1^4 \frac{1}{x^2}\,dx = \frac{1}{3}\cdot\frac{3}{4} = \boxed{\frac{1}{4}}$$

$$\int_1^4 \frac{1}{x^2}\,dx = \int_1^4 x^{-2}\,dx \underset{n=-2}{=} \frac{1}{-2+1}\cdot x^{-2+1}\Big|_1^4 = \frac{1}{-1}\cdot x^{-1}\Big|_1^4$$

$$= -1\cdot\frac{1}{x}\Big|_1^4$$

$$= -\frac{1}{4}-\left(-\frac{1}{1}\right) = -\frac{1}{4}+1 = \boxed{\frac{3}{4}}$$

- Mean Value Theorem for Integral.

If $f(x)$ is continuous on $[a,b]$, then there is some $c \in (a,b)$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Rmk: The above formula can be derived from MVT and FTC2.

eg5. For $f(x)=\frac{1}{x^2}$ on $[1,4]$ in eg.4, find the value of $c\in(1,4)$ such that $f(c) = f_{ave}$.

sln: From eg4, we know that $f_{ave} = \frac{1}{4}$

Then set $f(c) = \frac{1}{c^2} = \frac{1}{4}$. solve for c.

$$\Rightarrow 4 = c^2 \Rightarrow c = \pm 2.$$

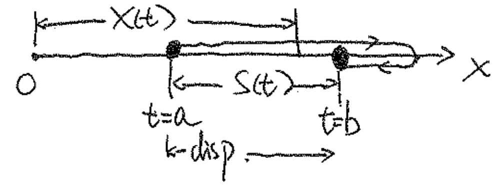
Only $\boxed{c=2}$ is in $(1,4)$.

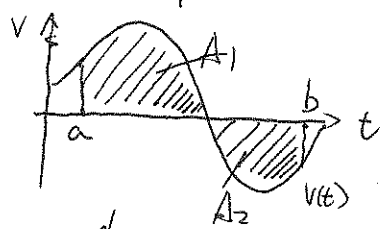
② Moving particle with velocity $V(t)$.

Position function $x(t) = \int V(t)\,dt$

☆ Displacement from $t=a$ to $t=b$ D.P. $= \int_a^b V(t)\,dt$

☆ Distance from $t=a$ to $t=b$ D.T. $= \int_a^b |V(t)|\,dt$





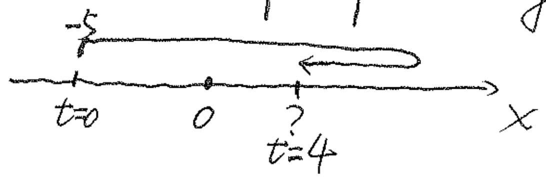
disp $= \int_a^b V\,dt = A_1 - A_2$

distance $= \int_a^b |V|\,dt = A_1 + A_2$

eg3. An object moves along a coordinate (to the right) line with velocity $V(t) = t(2-t)$.

Its initial position is 5 units to the left of the origin.

(a) Find its position after 4 seconds.



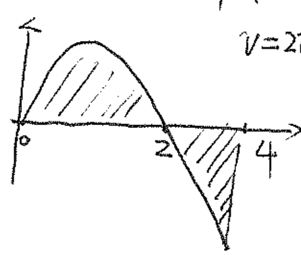
Displacement from $t=0$ to $t=4$.

$$\int_0^4 t(2-t)\,dt = \int_0^4 2t - t^2\,dt = t^2 - \frac{1}{3}t^3 \Big|_0^4 = (4^2 - \frac{1}{3}4^3) - 0 = 16 - \frac{64}{3}$$

Position = Initial position + Displacement $= -5 + 16 - \frac{64}{3} = \boxed{11 - \frac{64}{3}}$

☆ (b). Find the distance traveled from $t=0$ to $t=4$.

$$\text{Distance} = \int_0^4 |t(2-t)|\,dt$$



$v = 2t - t^2 > 0 \quad 0<t<2$

$V < 0 \quad 2<t<4$

$$= \int_0^2 |2t - t^2|\,dt + \int_2^4 |2t - t^2|\,dt$$

$$|V| = |2t - t^2| = \begin{cases} 2t - t^2 & 0<t<2 \\ t^2 - 2t & 2<t<4 \end{cases}$$

$$= \int_0^2 2t - t^2\,dt + \int_2^4 -(2t - t^2)\,dt$$

$$= t^2 - \frac{1}{3}t^3 \Big|_0^2 + (-t^2 + \frac{1}{3}t^3)\Big|_2^4$$

$$= 2^2 - \frac{1}{3}\cdot 2^3 - 0 + (-4^2 + \frac{1}{3}4^3) - (-2^2 + \frac{1}{3}2^3)$$

$$= \frac{48}{3} - 8 = \boxed{8}$$